

# Spectral stability of weak relaxation shock profiles

March 5, 2008

Corrado MASCIA

Dipartimento di Matematica “G. Castelnuovo”  
Sapienza – Università di Roma  
P.le Aldo Moro, 2 - 00185 Roma (ITALY)

Kevin ZUMBRUN

Mathematics Department  
Indiana University  
Bloomington, IN 47405-4301 (USA)

---

**Abstract:** Using a combination of Kawashima- and Goodman-type energy estimates, we establish spectral stability of general small-amplitude relaxation shocks of symmetric dissipative systems. This extends previous results obtained by Plaza and Zumbrun [9] by singular perturbation techniques under an additional technical assumption, namely, that the background equation be noncharacteristic with respect to the shock.

---

## 1. INTRODUCTION

Let us consider the one-dimensional hyperbolic system with relaxation

$$(1) \quad w_t + F(w)_x = Q(w)$$

for the unknown  $w = w(x, t) \in \mathbb{R}^N$ ,  $x \in \mathbb{R}$ ,  $t > 0$ . Here  $F \in C^2(\mathbb{R}^N; \mathbb{R}^N)$  is such that  $dF(w)$  has  $N$  real distinct eigenvalues for any state  $w$  under consideration and  $Q \in C^1(\mathbb{R}^N; \mathbb{R}^N)$  has the structure  $Q(w) = (0_n, q(w))$  where  $q \in C^1(\mathbb{R}^N; \mathbb{R}^r)$ ,  $r = N - n$ . Additionally, we assume the function  $q$  to have a relaxation structure: let  $w = (u, v) \in \mathcal{U} \times \mathcal{V} \subseteq \mathbb{R}^n \times \mathbb{R}^r$ ,

- i. there exists a  $C^1$  function  $v^* : \mathcal{U} \subset \mathbb{R}^n \rightarrow \mathbb{R}^r$  such that  $q(w) = 0$  in  $\mathcal{U} \times \mathcal{V}$  if and only if  $v = v^*(u)$  where  $w = (u, v) \in \mathbb{R}^n \times \mathbb{R}^r$ ;
- ii. for any  $u \in \mathcal{U}$ , all of the eigenvalues of  $d_u q(u, v^*(u))$  have negative real part.

As a consequence, it is natural to introduce the corresponding **relaxed** hyperbolic system of conservation laws, formally obtained by considering the first  $n$  equations of (1) and substituting the variable  $v$  with the equilibrium  $v = v^*(u)$

$$(2) \quad u_t + f^*(u)_x = 0 \quad \text{where} \quad f^*(u) := f(u, v^*(u)).$$

System (1) possesses smooth traveling wave solutions corresponding to shock waves of the relaxed system (2) at least in the small-amplitude case.<sup>1</sup> Existence of such special solutions has been given for specific models in the large-amplitude case — for example, the Broadwell model, [1] —, or for general relaxation system in the small-amplitude case, see [11, 7]. By changing frame, such travelling wave can be assumed, without loss of generality, as stationary solution of (1), i.e. solution of the form

$$(3) \quad W = W(x), \quad W(\pm\infty) = W_{\pm}.$$

where  $W_{\pm} = (u_{\pm}, v^*(u_{\pm}))$  with  $u_{\pm}$  denoting the state connected by the corresponding relaxed shock wave.

The next natural question to answer is whether such steady states are stable or unstable. The equation for the perturbation  $w := \tilde{w} - W$  is

$$w_t + (F(W + w) - F(W))_x = Q(W + w) - Q(W).$$

and the corresponding linearized equation is

$$(4) \quad w_t = \mathcal{L}w := -(\mathrm{d}F(W)w)' + \mathrm{d}Q(W)w.$$

Thus, the linearized eigenvalue equation is

$$(5) \quad (\lambda I - \mathcal{L})w = \lambda w + (\mathrm{d}F(W)w)' - \mathrm{d}Q(W)w = 0.$$

From now on, we consider equation (5) in the Sobolev space  $H^1$  and we say that  $\lambda \in \mathbb{C}$  is an eigenvalue of  $\mathcal{L}$  if there exists a function  $w \in H^1 \setminus \{0\}$ , such that (5) holds.<sup>2</sup>

By differentiating the equation satisfied by the profile  $W$ , we get

$$(\mathrm{d}F(W)W')' - \mathrm{d}Q(W)W' = 0.$$

In the noncharacteristic case, i.e.  $\mathrm{d}F(W_{\pm})$  invertible,  $W'$  decays exponentially fast to zero as  $|x| \rightarrow \infty$ . Thus,  $\lambda = 0$  is an eigenvalue of the linearized operator  $\mathcal{L}$ . Hence, instability is related to the presence of non-zero eigenvalues with non-negative real part.

---

<sup>1</sup>In all of the paper, we will not consider the case of nonsmooth relaxation shock profiles, i.e. exhibiting subshocks. This is not restrictive since the smallness assumption of the profile, usually, guarantees also smoothness.

<sup>2</sup>On the region  $\operatorname{Re} \lambda \geq 0$  and  $\lambda \neq 0$  that we will consider, and under our hypotheses (A1)–(A2) below,  $H^1$  spectrum agrees with  $L^p$  spectrum for any  $1 \leq p \leq \infty$ ; see [8].

*Definition 1.1.* The stationary solution  $W$  is **spectrally stable** in  $H^1$ , if for any  $\lambda \in \mathbb{C} \setminus \{0\}$ ,  $\operatorname{Re} \lambda \geq 0$ , whenever  $w \in H^1$  solves the resolvent equation (5) then  $w \equiv 0$ .

Results in [7, 8] show that, under additional assumptions, spectral stability implies both linear and nonlinear orbital stability. Hence, determining whether spectral stability holds or not is the key issue for determining nonlinear stability of relaxation shock profiles. The aim of the present paper is to prove a general result on spectral stability of relaxation shock profiles assuming smallness of  $\varepsilon := |W_+ - W_-|$ .

We divide the assumptions into three groups: **A**, **B** and **C**, referring, respectively, to the relaxation system (1), to the the relaxed system (2), to the relaxation shock profile (3). These are imposed on a small neighborhood  $\mathcal{W} \subset \mathbb{R}^n$  about an equilibrium point  $w_0 = (u_0, v_0)$ ,  $q(w_0) = 0$ , or equivalently  $v_0 = v_*(u_0)$ , and a neighborhood  $\mathcal{U}^*$  of  $u_0$  such that the graph of  $v_*$  over  $\mathcal{U}^*$  is contained in  $\mathcal{W}$ .

**A1.** *There exists a smooth function  $A_0 = A_0(W)$  from  $\mathcal{W} \subset \mathbb{R}^n$  to the set of real symmetric positive definite matrices such that  $(A_0 dF)(W)$  is symmetric for any  $W$  under consideration, and*

$$(6) \quad \operatorname{Re} \langle w, (A_0 dQ)(W_\pm) w \rangle \leq -c |\Pi_\pm w|_{L^2}^2,$$

for some  $c > 0$ , where  $\Pi_\pm w := dQ(W_\pm) w$ .

**A2.** *(Shizuta–Kawashima condition) There exists a smooth skew-symmetric matrix-valued function  $K$ , depending on  $dF, dQ$  and  $A_0$  such that*

$$(7) \quad \operatorname{Re} (K dF - A_0 dQ) > 0.$$

**B1.** *There exists a smooth function  $a_0 = a_0(u)$  from  $\mathcal{U}^* \subset \mathbb{R}^p$  to the set of real symmetric positive definite matrices such that  $a_0 df^*(u)$  is symmetric for any  $u$  under consideration and  $a_0 b_0$  is positive semidefinite,  $\operatorname{Re} a_0 b_0 \geq 0$ , where  $b_0$  as defined in (3) is the associated Chapman–Enskog viscosity.*

**B2.** *(reduced Shizuta–Kawashima condition) There exists a smooth skew-symmetric matrix-valued function  $k$ , depending on  $df^*, b_0$  and  $a_0$  such that*

$$(8) \quad \operatorname{Re} (k df^* - a_0 b_0) > 0.$$

**B3.** *(Simplicity, genuine nonlinearity of principle equilibrium characteristic) There exists  $c_0 > 0$  such that there is a single eigenvalue  $\alpha_0(u)$  of  $df^*(u)$  that has absolute*

value  $< c_0$  on  $\mathcal{U}^*$ , with all others of absolute value  $\geq 2c_0$ . Moreover,

$$(9) \quad d\alpha_0(u) \cdot r_0(u) =: \eta(u) \neq 0$$

on  $\mathcal{U}^*$ , where  $r_0(u)$  denotes the unit right eigenvector associated with  $\alpha_0(u)$ .

C. There exists  $C > 0$  such that for any  $x \in \mathbb{R}$  there hold

$$(10) \quad |W'|_{L^\infty} \leq C|W_+ - W_-|^2, \quad |W''(x)| \leq C|W_+ - W_-| |W'(x)|,$$

and

$$(11) \quad \left| \frac{W'}{|W'|} + \operatorname{sgn}(\eta)r_0 \right| \leq C|W_+ - W_-|^2.$$

*Remark 1.2.* The apparently restrictive A1–A2, B1–B2 in fact all follow from the standard assumptions that (i) there exist a positive definite symmetrizer  $A_0$ ,  $A_0 dF$  symmetric, that *at equilibrium points* simultaneously symmetrizes  $dQ$ ,  $A_0 dQ$  symmetric (weak simultaneous symmetrizability), and (ii) at equilibrium points, no eigenvector of  $dF$  is in the kernel of  $dQ$  (genuine coupling); see Lemma A.1, Appendix A. These two assumptions hold quite generally in applications, in particular for discrete kinetic equations and moment closure systems [10]. Assumption B3 is standard and easily checked.

*Remark 1.3.* Assumption C is satisfied for a family of profiles near  $w_0$  if:

- (i)  $dF$  is invertible and  $\alpha_0(u_0) = 0$  (see Appendix of [7]).
- (ii)  $dF$  is constant,  $\alpha_0(u_0) = 0$ , and dimension  $N$  bounded, e.g. in the case of discrete kinetic models with upper bound on the number of modes (see [8]). It has been shown to hold also for the infinite-dimensional case of the Boltzmann equation [6].

With these assumptions, our main result is as follows.

**Theorem 1.4** (Spectral stability). *Under assumptions A1–A2, B1–B2–B3, and C, for  $\varepsilon := |W_+ - W_-|$  sufficiently small, the relaxation shock  $W$  is spectrally stable.*

As the argument is somewhat complicated, it may be helpful to outline here the structure of the proof. We start by carrying out the following by-now-standard “Kawashima-type” energy estimates on the relaxation system.

**Proposition 1.5.** *Assume hypothesis A1-A2 and C. Let  $\lambda \in \mathbb{C}$  such that  $\operatorname{Re} \lambda \geq 0$  and let  $w$  be a solution of (5). Then for  $\varepsilon$  sufficiently small, there hold:*

$$(12) \quad \operatorname{Re} \lambda |w|_{L^2}^2 + |\Pi w|_{L^2}^2 + |w'|_{L^2}^2 \leq C\varepsilon^2 |u|_{L^2}^2 \quad (\text{Kawashima estimate})$$

$$(13) \quad \operatorname{Re} \lambda \leq C\varepsilon^2, \quad |\operatorname{Im} \lambda| \leq C\varepsilon.$$

where  $\Pi := \Pi_+ + \Pi_-$ .

Evidently, it remains only to obtain estimates on the equilibrium variable  $|u|_{L^2}^2$ . To this end, we carry out an approximate Chapman–Enskog expansion, keeping track of error terms, to obtain an effective viscous system for  $u$  of the same symmetric dissipative type, but with error terms coming from higher derivatives. Applying Goodman-type energy estimates to the integrated version of this reduced system, following [5], we obtain the desired bounds on  $|u|_{L^2}^2$  modulo errors consisting of higher-derivative and dissipative terms (denoted by  $\hat{v}$  in Section 3). Observing that these, by (12), may be absorbed in lower-order and equilibrium terms, we are done.

More precisely, we establish the following bounds on the reduced system. Here we use the following notation for the  $W'$ -weighted  $L^2$ -norm

$$|z|_{W'} := \left| \sqrt{|W'|} z \right|_{L^2} = \left( \int_{\mathbb{R}} |z|^2 |W'| (x) dx \right)^{1/2}$$

The space of functions with bounded  $|\cdot|_{W'}$  will be denoted by  $L_{W'}^2$ . Since  $W'$  is bounded, there holds  $|z|_{W'} \leq C |z|_{L^2}$  for some  $C > 0$  (a natural choice is  $C := |W'|_{L^\infty}^{1/2}$ ). Hence  $L^2 \subset L_{W'}^2$  with continuous injection. The opposite inequality is false since  $W'$  decays (exponentially fast) to zero as  $|x| \rightarrow \infty$ .

**Proposition 1.6.** *Assume hypothesis B1-B2 and C. Let  $z$  be defined as follows*

$$z(x) := \int_{-\infty}^x u(y) dy.$$

*Then, for  $\varepsilon$  sufficiently small, there holds*

$$(14) \quad \operatorname{Re} \lambda |z|_{L^2}^2 + |u|_{L^2}^2 \leq C |z|_{W'}^2.$$

Thanks to (14), it is sufficient to control the weighted norm  $|z|_{W'}$ .

**Proposition 1.7.** *Under assumptions B1–B2–B3 and C, for  $\varepsilon$  sufficiently small, there holds:*

$$(15) \quad \operatorname{Re} \lambda |z|_{L^2}^2 + |z|_{W'}^2 \leq C(\varepsilon |u|_{L^2}^2 + \varepsilon |\Pi w|_{L^2}^2 + \varepsilon^{-1} |w'|_{L^2}^2) \quad (\text{Goodman estimate})$$

where  $\Pi := \Pi_+ + \Pi_-$ .

Theorem 1.4 is an immediate consequence of Propositions 1.5, 1.6 and 1.7.

*Proof of Theorem 1.4.* Combining (12) and (15), we obtain

$$|z|_{w'}^2 \leq C \varepsilon |u|_{L^2}^2.$$

With (14), this gives

$$|u|_{L^2}^2 \leq C |z|_{w'}^2 \leq C \varepsilon^2 |u|_{L^2}^2,$$

showing that, if  $\varepsilon = |W_+ - W_-|$  is small enough,  $w \equiv 0$ .  $\square$

**Discussion and open problems.** We remark briefly on the setting of these results. Small-amplitude existence and stability were shown in [11, 7] and [9] under the additional noncharacteristicity assumption  $\det dF \neq 0$ , or, equivalently, the condition that characteristic speeds of the background system do not vanish relative to the shock speed. This hypothesis suffices to treat simple model problems such as the Broadwell or Jin–Xin equations. However, as discussed in [8], it is unrealistic for models derived by discretization or moment closure from kinetic equations, since these may possess characteristics of any speed. Thus, it is highly desirable to remove this technical hypothesis, as we do here. The combination of Goodman- and Kawashima-type energy estimates was used in [5] to treat stability of viscous shock profiles for systems with real viscosity. A similar, but more complicated argument combining these ingredients was used in [6] to treat stability of Boltzmann profiles. These results motivate the present analysis, which essentially interpolates between the two.

Interesting open problems are verification of linearized and nonlinear stability in the same setting, assuming spectral stability, and the direct verification of  $\mathbf{C}$  using stability estimates together with known bounds on the profile for the reduced system following the philosophy set out in [6].

**Notations and (very) basic tools.** Given  $w_1, w_2 : \mathbb{R} \rightarrow \mathbb{C}^n$ , we denote by  $\langle w_1, w_2 \rangle$  the scalar product defined as follows

$$\langle w_1, w_2 \rangle := \int_{\mathbb{R}} \overline{w_1(x)} \cdot w_2(x) dx$$

where  $\bar{w}$  denotes the complex conjugate vector of  $w$ . Given  $A$ ,  $n \times n$  matrix with complex entries, there holds

$$\operatorname{Re} \langle w, Aw \rangle = \frac{1}{2} \left( \langle w, Aw \rangle + \overline{\langle w, Aw \rangle} \right) = \langle w, A^* w \rangle$$

where  $A^* := (A + \overline{A^t})/2$ .

If  $S : \mathbb{R} \rightarrow \mathbb{R}^{n \times n}$  is such that  $S(x)$  is symmetric for any  $x$ , then

$$\langle w, S w' \rangle = \int_{\mathbb{R}} \bar{w} \cdot S w' dx = - \int_{\mathbb{R}} \overline{S w'} \cdot w dx - \int_{\mathbb{R}} \bar{w} \cdot S' w dx = - \overline{\langle w, S w' \rangle} - \langle w, S' w \rangle$$

Hence

$$(16) \quad \operatorname{Re} \langle w, S w' \rangle = -\frac{1}{2} \langle w, S' w \rangle.$$

Similarly, if  $K$  is skew-symmetric, then

$$(17) \quad \operatorname{Im} \langle w', K w \rangle = -\frac{1}{2} \langle w, K' w \rangle$$

In particular, if  $K$  is constant,  $\langle w', K w \rangle$  is a real number.

From here on, we will denote with  $O(1)$  any function of  $x, W$  and  $\lambda$ , locally bounded in  $\{(x, W, \lambda) : \operatorname{Re} \lambda \geq 0\}$ . As a consequence, given the functions  $f, g \in L^2$  and  $h \in L^2_{w'}$ , the following estimates hold

$$(18) \quad |\langle f, O(1) W' g \rangle| \leq C |W'|_{L^\infty} \left( \eta |f|_{L^2}^2 + \eta^{-1} |g|_{L^2}^2 \right)$$

$$(19) \quad |\langle h, O(1) W' g \rangle| \leq C \left( \eta |h|_{L^2_{w'}}^2 + \eta^{-1} |W'|_{L^\infty} |g|_{L^2}^2 \right)$$

where  $\eta$  is any strictly positive constant and  $C$  is a constant independent on  $\eta$ .

## 2. ESTIMATES FOR THE FULL SYSTEM

**Lemma 2.1.** *Let  $\varepsilon := |W_+ - W_-|$  and assume hypothesis **A1** and **C**. Let  $\lambda \in \mathbb{C}$  such that  $\operatorname{Re} \lambda \geq 0$  let  $w$  be a solution of (5) and let  $K$  be any constant skew-symmetric matrix. Then for  $\varepsilon, \eta > 0$  both sufficiently small, there hold*

$$(20) \quad \operatorname{Re} \lambda |w|_{L^2}^2 + |\Pi w|_{L^2}^2 \leq C \varepsilon^2 |w|_{L^2}^2,$$

$$(21) \quad \operatorname{Re} \lambda |w'|_{L^2}^2 - \operatorname{Re} \langle w', A_0 dQ w' \rangle \leq C \varepsilon^2 |w|_{H^1}^2;$$

$$(22) \quad \operatorname{Re} \langle w', K dF w' \rangle \leq C \left( \varepsilon^2 |w|_{L^2}^2 + \eta^{-1} |\Pi w|_{L^2}^2 + (\varepsilon^2 + \eta) |w'|_{L^2}^2 \right)$$

where  $\Pi := \Pi_+ + \Pi_-$  and  $C$  denotes a constant independent on  $\varepsilon$  and  $\eta$ .

*Proof.* Taking the scalar product of  $A_0(W) w$  against (5), we obtain

$$(23) \quad \lambda \langle A_0 w, w \rangle + \langle A_0 w, (dF w)' \rangle - \langle A_0 w, dQ w \rangle = 0.$$

Hence, using (16), we get

$$\operatorname{Re} \lambda \langle A_0 w, w \rangle - \operatorname{Re} \langle w, A_0 dQ w \rangle \leq -\operatorname{Re} \langle A_0 w, d^2 F W' w \rangle + \frac{1}{2} \operatorname{Re} \langle w, d(A_0 dF) W' w \rangle.$$

Since  $A_0$  is positive definite, there holds for some  $C > 0$

$$(24) \quad \operatorname{Re} \lambda |w|_{L^2}^2 - \operatorname{Re} \langle w, A_0 dQ w \rangle \leq C |W'|_{L^\infty} |w|_{L^2}^2.$$

Let us set

$$\Phi(W) := \frac{|W - W_+|}{|W_+ - W_-|} (A_0 dQ)(W_-) + \frac{|W - W_-|}{|W_+ - W_-|} (A_0 dQ)(W_+).$$

Then there holds, for some  $C > 0$ ,

$$|\Phi(W) - (A_0 dQ)(W)| \leq C |W - W_-| |W - W_+|.$$

Therefore

$$\operatorname{Re} \langle w, A_0 dQ w \rangle \leq \operatorname{Re} \langle w, \Phi(W) w \rangle + C |W - W_-|_{L^\infty} |W - W_+|_{L^\infty} |w|_{L^2}^2$$

Thanks to (6), we get

$$\operatorname{Re} \langle w, A_0 dQ w \rangle \leq -c |\Pi w|_{L^2}^2 + C |W - W_-|_{L^\infty} |W - W_+|_{L^\infty} |w|_{L^2}^2$$

for some  $C, c > 0$ . Hence, using (24), we obtain

$$(25) \quad \operatorname{Re} \lambda |w|_{L^2}^2 + |\Pi w|_{L^2}^2 \leq C (|W - W_+|_{L^\infty} |W - W_-|_{L^\infty} + |W'|_{L^\infty}) |w|_{L^2}^2.$$

In term of  $\varepsilon$ , we get the *0-th order Friedrichs estimate* (20).

Differentiating (5) with respect to  $x$ ,

$$(26) \quad \lambda w' + (dF(W)w)'' - (dQ(W)w)' = 0.$$

Taking the scalar product of  $A_0(W) w'$  against (26), we get

$$(27) \quad \lambda \langle A_0 w', w' \rangle - \langle w', A_0 dQ w' \rangle = -\langle A_0 w', (dF w)'' \rangle + \langle A_0 w', d^2 Q W' w \rangle.$$

Since

$$\begin{aligned} \langle A_0 w', (dF w)'' \rangle &= \langle A_0 w', d^3 F W' W' w \rangle + \langle A_0 w', d^2 F W'' w \rangle \\ &\quad + 2 \langle A_0 w', d^2 F W' w' \rangle + \langle w', A_0 dF w'' \rangle, \end{aligned}$$

taking the real part and using (16), we obtain

$$\operatorname{Re} \langle A_0 w', (dF w)'' \rangle = \langle w', O(1) W' w \rangle + \langle w', O(1) W'' w \rangle$$



$$+\langle w', O(1)W' w' \rangle - \frac{1}{2}\langle w', d(A_0 dF) W' w' \rangle.$$

Hence, the following estimates holds

$$\begin{aligned} |\operatorname{Re} \langle A_0 w', (dF w)'' \rangle| &\leq C(|W'|_{L^\infty} + |W''|_{L^\infty}) |w|_{H^1}^2 \\ |\operatorname{Re} \langle A_0 w', d^2 Q W' w \rangle| &\leq C |W'|_{L^\infty} |w|_{H^1}^2. \end{aligned}$$

Therefore, from (27), using (10), we deduce

$$\operatorname{Re} \lambda |w'|_{L^2}^2 - \operatorname{Re} \langle w', A_0 dQ w' \rangle \leq C |W'|_{L^\infty} |w|_{H^1}^2.$$

Thus in term of  $\varepsilon$ , we obtain the *1-st order Friedrichs estimate* (21).

Now, let  $K$  be any constant skew-symmetric matrix. Applying  $K$  to the resolvent equation (5) and multiplying by  $w'$ , we get

$$\lambda \langle w', Kw \rangle + \langle w', K(dF w)' \rangle - \langle w', K dQ w \rangle = 0.$$

Taking the real parts and rearranging the terms, we obtain

$$\operatorname{Re} \langle w', K dF w' \rangle = -\operatorname{Re} (\lambda \langle w', Kw \rangle) - \operatorname{Re} \langle w', K d^2 F W' w \rangle + \operatorname{Re} \langle w', K dQ w \rangle.$$

Hence, thanks to (17), there holds  $\operatorname{Im} \langle w', Kw \rangle = 0$ , since  $K$  is constant. Therefore, for  $\operatorname{Re} \lambda \geq 0$ , we obtain

$$|\operatorname{Re} (\lambda \langle w', Kw \rangle)| = \operatorname{Re} \lambda |\operatorname{Re} \langle w', Kw \rangle| \leq C \operatorname{Re} \lambda |w|_{H^1}^2.$$

Let us set

$$\Psi(W) := \frac{|W - W_+|}{|W_+ - W_-|} K dQ(W_-) + \frac{|W - W_-|}{|W_+ - W_-|} K dQ(W_+).$$

Then there holds, for some  $C > 0$ ,

$$|\Psi(W) - K dQ(W)| \leq C |W - W_-| |W - W_+|.$$

Therefore

$$\operatorname{Re} \langle w', K dQ w \rangle \leq \operatorname{Re} \langle w', \Psi(W) w \rangle + C |W - W_-|_{L^\infty} |W - W_+|_{L^\infty} |w|_{H^1}^2$$

Thanks to (6), we get

$$\operatorname{Re} \langle w', K dQ w \rangle \leq C (\eta^{-1} |\Pi w|_{L^2}^2 + \eta |w'|_{L^2}^2) + C |W - W_-|_{L^\infty} |W - W_+|_{L^\infty} |w|_{L^2}^2$$

where  $\eta > 0$  is a positive constant to be chosen later on (small enough). Hence, the following estimate holds

$$\begin{aligned} \operatorname{Re} \langle w', K dF w' \rangle &= C \{ (\operatorname{Re} \lambda + |W'|_{L^\infty} + |W - W_-|_{L^\infty} |W - W_+|_{L^\infty}) |w|_{H^1}^2 \\ &\quad + \eta^{-1} |\Pi w|_{L^2}^2 + \eta |w'|_{L^2}^2 \} \end{aligned}$$

By (20),  $\operatorname{Re} \lambda \leq C \varepsilon^2$ , hence, in term of  $\varepsilon$ , we get (22). This concludes the proof of Lemma 2.1.  $\square$

Assuming, in addition, hypothesis **A2**, we prove estimates (12) and (13).

*Proof of Proposition 1.5.* Thanks to the small amplitude assumption, it is possible to choose  $K = K(W_+)$  constant in the Shizuta–Kawashima condition (7), since this is an open condition so persists under small perturbations. Hence, summing estimates (20)–(21) with (22), we obtain

$$\operatorname{Re} \lambda |w|_{L^2}^2 + |\Pi w|_{L^2}^2 + |w'|_{L^2}^2 \leq C \varepsilon^2 |w|_{H^1}^2 + C(1 + \eta^{-1}) \varepsilon^2 |w|_{L^2}^2 + C(\varepsilon^2 + \eta) |w'|_{L^2}^2$$

which yields

$$\operatorname{Re} \lambda |w|_{L^2}^2 + |\Pi w|_{L^2}^2 + |w'|_{L^2}^2 \leq C \varepsilon^2 |w|_{L^2}^2$$

for  $\varepsilon$  and  $\eta$  sufficiently small. Since  $|w|_{L^2}^2 \leq C(|u|_{L^2}^2 + |\Pi w|_{L^2}^2)$  we get the estimate (12) for  $\varepsilon$  small.

Estimate (20) implies the bound on the real part of the eigenvalue  $\lambda$ . Taking the imaginary part of (23),

$$\operatorname{Im} (\lambda \langle A_0 w, w \rangle) = -\operatorname{Im} \langle A_0 w, d^2 F W' w \rangle - \operatorname{Im} \langle A_0 w, dF w' \rangle + \operatorname{Im} \langle A_0 w, dQ w \rangle.$$

Hence, for  $\eta > 0$  to be chosen,

$$|\operatorname{Im} \lambda| |w|_{L^2}^2 \leq C \left( |W'|_{L^\infty} |w|_{L^2}^2 + \eta |w|_{L^2}^2 + \eta^{-1} |w'|_{L^2}^2 + \eta^{-1} |\Pi w|_{L^2}^2 \right).$$

Thanks to (10) and (12), we get

$$|\operatorname{Im} \lambda| |w|_{L^2}^2 \leq C (\varepsilon^2 + \eta + \eta^{-1} \varepsilon^2) |w|_{L^2}^2.$$

Thus, choosing  $\eta = \varepsilon$ , we obtain the result for  $\varepsilon$  small enough.  $\square$

### 3. THE REDUCED SYSTEM FOR THE CONSERVED INTEGRATED VARIABLES

As stressed in the Introduction, the next step consists in estimating the conserved densities  $u$  in term of an appropriate weighted  $L^2$ –norm of the conserved quantities  $z$ , defined by

$$(28) \quad z(x) := \int_{-\infty}^x u(y) dy.$$

The first step is to deduce a balance law satisfied by the variable  $z$  with source terms depending on  $\Pi w$  and  $w'$ .

Setting

$$dF := \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \quad \text{and} \quad dQ := \begin{pmatrix} 0 & 0 \\ q_1 & q_2 \end{pmatrix},$$

equation (5) can be rewritten as

$$\begin{cases} \lambda u + (A_{11} u + A_{12} v)' = 0, \\ \lambda v + (A_{21} u + A_{22} v)' - q_1 u - q_2 v = 0. \end{cases}$$

Let  $\hat{v} := q_2^{-1} q_1 u + v$ . In particular,  $c_1 |\hat{v}|_{L^2} \leq |\Pi w|_{L^2} \leq c_2 |\hat{v}|_{L^2}$  for some  $c_1, c_2 > 0$ . Hence, in the following, we consider the variable  $\hat{v}$  in place of  $\Pi w$ .

Then the couple  $(u, \hat{v})$  satisfies

$$\begin{cases} \lambda u + (a u + A_{12} \hat{v})' = 0, \\ (\lambda I_r - q_2) \hat{v} + (c u + A_{22} \hat{v})' + q_2^{-1} q_1 (a u + A_{12} \hat{v})' = 0 \end{cases}$$

where

$$a := A_{11} - A_{12} q_2^{-1} q_1, \quad c := A_{21} - A_{22} q_2^{-1} q_1.$$

With  $z$  defined in (28), we can write the above system as

$$\begin{cases} \lambda z + a z' + A_{12} \hat{v} = 0, \\ (\lambda I_r - q_2) \hat{v} + (c z' + A_{22} \hat{v})' + q_2^{-1} q_1 (a z' + A_{12} \hat{v})' = 0 \end{cases}$$

Next the idea is to obtain an expression for  $\hat{v}$  from the second equation and inserting it in the first one, in order to obtain a reduced system of viscous conservation laws with source terms. Since we want to derive energy estimates, it is useful to change variables in the first equation in order to symmetrize the term containing the first order derivative.

Let  $a_0$  be a symmetric and positive definite matrix such that  $a_0 a$  is symmetric, as in assumption **B1**. Let  $\tilde{z} := a_0^{1/2} z$ . The new variable  $\tilde{z}$  and the variable  $\hat{v}$  satisfy

$$\begin{cases} \lambda \tilde{z} + a_0^{1/2} a (a_0^{-1/2} \tilde{z})' + a_0^{1/2} A_{12} \hat{v} = 0, \\ (\lambda I_r - q_2) \hat{v} + (c (a_0^{-1/2} \tilde{z})' + A_{22} \hat{v})' + q_2^{-1} q_1 (a (a_0^{-1/2} \tilde{z})' + A_{12} \hat{v})' = 0 \end{cases}$$

Hence

$$\begin{cases} \lambda \tilde{z} + \tilde{a} \tilde{z}' + a_0^{1/2} A_{12} \hat{v} = O(1) W' \tilde{z}', \\ (\lambda I_r - q_2) \hat{v} + (c a_0^{-1/2} \tilde{z}' + O(1) W' \tilde{z} + O(1) \hat{v})' \\ \quad + q_2^{-1} q_1 (a a_0^{-1/2} \tilde{z}' + O(1) W' \tilde{z} + O(1) \hat{v})' = 0 \end{cases}$$

where the matrix  $\tilde{a} := a_0^{1/2} a a_0^{-1/2}$  is symmetric. From the second equation, using (10), we get

$$(29) \quad \hat{v} = -(\lambda I_r - q_2)^{-1} (c + q_2^{-1} q_1 a) a_0^{-1/2} \tilde{z}'' + O(1) W' (\tilde{z}' + \varepsilon \tilde{z} + \hat{v}) + O(1) \hat{v}',$$

or, equivalently, using the  $O(1)$  notation,

$$(30) \quad \hat{v} = O(1)W'(\tilde{z}' + \varepsilon\tilde{z} + \hat{v}) + O(1)w'.$$

Plugging (29) in the equation satisfied by  $\tilde{z}$ , we get

$$(31) \quad \lambda \tilde{z} + \tilde{a} \tilde{z}' - \tilde{b} \tilde{z}'' = O(1)W'(\tilde{z}' + \varepsilon\tilde{z} + \hat{v}) + O(1)\hat{v}'$$

where

$$b := A_{12}(\lambda I_r - q_2)^{-1}(c + q_2^{-1}q_1a) \quad \text{and} \quad \tilde{b} := a_0^{1/2} b a_0^{-1/2}.$$

Estimate on  $\tilde{z}$  will be obtained by multiplying (31) by  $z$  and integrating. With the present form, we would obtain a "bad" term  $\langle \tilde{z}, O(1)\hat{v}' \rangle$ . For this reason, it is useful to use the relation (30) to obtain the following new version of (31)

$$\lambda \tilde{z} + \tilde{a} \tilde{z}' - \tilde{b} \tilde{z}'' = O(1)W'(\tilde{z}' + \varepsilon\tilde{z} + \hat{v}) + O(1)(O(1)W'(\tilde{z}' + \varepsilon\tilde{z} + \hat{v}) + O(1)w')'.$$

For  $\lambda = 0$ , the term  $b$  represents the viscosity term given by the Chapman–Enskog expansion for the relaxation system. Hence, it is significant to decompose  $b$  as follows

$$b = b_0 + \lambda b_1$$

where matrices  $b_0$  and  $b_1$  are given by

$$b_0 := -A_{12}q_2^{-1}(c + q_2^{-1}q_1a), \quad b_1 := A_{12}(\lambda - q_2)^{-1}q_2^{-1}(c + q_2^{-1}q_1a).$$

Hence we get the following equation satisfied by the variable  $\tilde{z}$

$$\lambda \tilde{z} + \tilde{a} \tilde{z}' - \tilde{b}_0 \tilde{z}'' = \lambda \tilde{b}_1 \tilde{z}'' + \Theta.$$

where  $\tilde{b}_i := a_0^{1/2} b_i a_0^{-1/2}$  for  $i = 0, 1$  and  $\Theta$  is appropriately defined. By assumptions B1. on the reduced system, the matrix  $\tilde{b}_0$  is symmetric and positive semidefinite.

From now on, we drop the tildas for shortness and consider the following equation

$$(32) \quad \lambda z + a z' - b_0 z'' = \lambda b_1 z'' + \Theta_1 + \Theta_2'.$$

where  $a$  is a symmetric matrix,  $b_0$  is a symmetric, positive semidefinite matrix, and

$$(33) \quad \begin{cases} \Theta_1 := O(1)W'(z' + \varepsilon z + \hat{v} + w') \\ \Theta_2 := O(1)W'(z' + \varepsilon z + \hat{v}) + O(1)w' \end{cases}$$

**Lemma 3.1.** *Assume hypothesis C. Let  $\lambda \in \mathbb{C}$  satisfying (13), let  $z$  be a solution (32) with  $a$  symmetric,  $b_0$  symmetric, positive semidefinite and  $\Theta_1, \Theta_2$  given in (33) and let  $k$  be a smooth function from  $\mathbb{R}^n$  to the set of real skew-symmetric matrices.*

Then, for  $\varepsilon, \eta > 0$  sufficiently small, the following estimates hold:

$$\begin{aligned} \operatorname{Re} \lambda |z|_{L^2}^2 + \langle z', b_0 z' \rangle &\leq C(|z|_{W'}^2 + (\varepsilon + \eta)|z'|_{L^2}^2 + \varepsilon^2|\hat{v}|_{L^2}^2 + \eta^{-1}|w'|_{L^2}^2) \\ \operatorname{Re} \langle z', k a z' \rangle &\leq C(\eta \operatorname{Re} \lambda |z|_{L^2}^2 + \varepsilon|z|_{W'}^2 + (\eta^{-1}\varepsilon^2 + \eta)|z'|_{L^2}^2 + \varepsilon^2|\hat{v}|_{L^2}^2 + \eta^{-1}|w'|_{L^2}^2) \end{aligned}$$

*Proof.* Taking the real part of the scalar product of  $z$  against (32), we get

$$\operatorname{Re} \lambda |z|_{L^2}^2 - \frac{1}{2} \langle z, da W' z \rangle - \operatorname{Re} \langle z, b_0 z'' \rangle = \operatorname{Re} \langle z, \lambda b_1 z'' \rangle + \operatorname{Re} \langle z, \Theta \rangle.$$

having used the symmetry of  $a$ . Since

$$\begin{aligned} \operatorname{Re} \langle z, b_0 z'' \rangle &= -\operatorname{Re} \langle z', b_0 z' \rangle - \operatorname{Re} \langle z, db_0 W' z' \rangle \\ &\leq -\operatorname{Re} \langle z', b_0 z' \rangle + C(|z|_{W'}^2 + |W'|_{L^\infty} |z'|_{L^2}^2). \end{aligned}$$

we obtain, thanks to (10),

$$\operatorname{Re} \lambda |z|_{L^2}^2 + \operatorname{Re} \langle z', b_0 z' \rangle \leq C(|z|_{W'}^2 + \varepsilon^2|z'|_{L^2}^2 + \operatorname{Re} \langle z, \lambda b_1 z'' \rangle + \operatorname{Re} \langle z, \Theta \rangle).$$

The term containing  $b_1$  can be easily estimated by

$$\begin{aligned} |\operatorname{Re} \langle z, \lambda b_1 z'' \rangle| &\leq |\lambda| (|\langle z', b_1 z' \rangle| + |\langle z, db_0 W' z' \rangle|) \\ &\leq C|\lambda| \left( |z'|_{L^2}^2 + |z|_{W'}^2 + |W'|_{L^\infty} |z'|_{L^2}^2 \right). \end{aligned}$$

Taking in account (10) and (13), we get

$$|\operatorname{Re} \langle z, \lambda b_1 z'' \rangle| \leq C\varepsilon \left( |z'|_{L^2}^2 + |z|_{W'}^2 \right).$$

Therefore, we obtain

$$(34) \quad \operatorname{Re} \lambda |z|_{L^2}^2 + \operatorname{Re} \langle z', b_0 z' \rangle \leq C(|z|_{W'}^2 + \varepsilon|z'|_{L^2}^2 + \operatorname{Re} \langle z, \Theta \rangle).$$

It remains to deal with the term with  $\Theta$ . For what concerns  $\Theta_1$ , using (19) (with  $\eta = 1$ ) and (10), we have

$$|\operatorname{Re} \langle z, \Theta_1 \rangle| \leq C|z|_{W'}^2 + C\varepsilon^2(|z'|_{L^2}^2 + |\hat{v}|_{L^2}^2 + |w'|_{L^2}^2)$$

The term with  $\Theta_2$  can be dealt with integrating by parts

$$|\operatorname{Re} \langle z, \Theta'_2 \rangle| = |\operatorname{Re} \langle z', \Theta_2 \rangle| \leq C\varepsilon|z|_{W'}^2 + C(\varepsilon^2 + \eta)|z'|_{L^2}^2 + C\varepsilon^2|\hat{v}|_{L^2}^2 + C\eta^{-1}|w'|_{L^2}^2$$

where  $\eta$  is any positive constant and  $C$  is independent on  $\eta$ . Inserting the last three estimates in (34), we get i. in Lemma 3.1.

Applying  $k$  to (32),  $k$  as defined in B2, and taking the  $L^2$  scalar product against  $z'$ , we get

$$(35) \quad \operatorname{Re} \langle z', k a z' \rangle = -\operatorname{Re} \lambda \langle z', k z \rangle + \operatorname{Re} \langle z', k b z'' \rangle + \operatorname{Re} \langle z', k \Theta \rangle.$$

Using the eigenvalue estimate on  $\operatorname{Re} \lambda$ , stated in Proposition 1.5, we obtain

$$\operatorname{Re} \langle z', k a z' \rangle \leq C \left( \eta |\operatorname{Re} \lambda| |z|_{L^2}^2 + (\eta^{-1} \varepsilon^2 + \eta) |z'|_{L^2}^2 + \eta^{-1} |z''|_{L^2}^2 \right) + \operatorname{Re} \langle z', k \Theta \rangle$$

Finally, using once more (10), for the term with  $\Theta$  there hold

$$\begin{aligned} |\operatorname{Re} \langle z', k \Theta_1 \rangle| &\leq C \varepsilon |z|_{W'}^2 + C \varepsilon^2 \left( |z'|_{L^2}^2 + |\hat{v}|_{L^2}^2 + |w'|_{L^2}^2 \right) \\ |\operatorname{Re} \langle z', k \Theta'_2 \rangle| &\leq |\operatorname{Re} \langle z'', k \Theta_2 \rangle| + |\operatorname{Re} \langle z', O(1) W' \Theta_2 \rangle| \\ &\leq C \varepsilon |z|_{W'}^2 + C \varepsilon^2 \left( |z'|_{L^2}^2 + |\hat{v}|_{L^2}^2 \right) + C |w'|_{L^2}^2. \end{aligned}$$

Collecting all of these estimates, we complete the proof of Lemma 3.1.  $\square$

*Proof of Proposition 1.6.* Choosing  $\eta = \varepsilon$  and summing up the estimates in Lemma 3.1, we obtain, for  $\varepsilon$  small enough,

$$\operatorname{Re} \lambda |z|_{L^2}^2 + |z'|_{L^2}^2 \leq C \left( |z|_{W'}^2 + \varepsilon^2 |\hat{v}|_{L^2}^2 + \varepsilon^{-1} |w'|_{L^2}^2 \right)$$

Using (in place of a first-order Friedrichs estimate) the bound

$$|\hat{v}|_{L^2}^2 + |w'|_{L^2}^2 \leq C \left( |\Pi v|_{L^2}^2 + |w'|_{L^2}^2 \right) \leq C \varepsilon^2 |u|_{L^2}^2$$

obtained in Proposition 1.5, we get

$$\operatorname{Re} \lambda |z|_{L^2}^2 + |u|_{L^2}^2 \leq C |z|_{W'}^2 + C \varepsilon |u|_{L^2}^2.$$

Hence estimate (14) holds for  $\varepsilon$  small.  $\square$

The reduced Kawashima estimate (14) shows that it is possible to bound the  $L^2$  estimate of  $u$  in term of  $|z|_{W'}$ . If we are able to prove a Poincaré-like inequality and bound the weighted norm  $|z|_{W'}$  by small multiples of the  $L^2$  norm of  $u$  and higher derivatives, we are done. This we can accomplish by changing variables in an appropriate way and applying a weighted energy method in the spirit of Goodman [4, 5, 6].

**Lemma 3.2** ([5]). *Let  $a = a(W)$  and  $b = b(W)$  be symmetric matrices,  $b(W) \geq 0$ , with one eigenvalue  $\alpha_0$  of  $a$  close to zero and the others strictly negative or positive (and uniformly separated from  $\alpha_0$ ). Then, there exist smooth, real matrix-valued functions  $r = r(x)$ ,  $\ell = \ell(x)$ ,  $\ell(x) r(x) = I$  for any  $x$ , satisfying, for some  $C, c > 0$ ,*

$$\begin{aligned} (\ell r')_{pp} &= 0, & |\ell'|, |r'| &\leq C |W'|; \\ (36) \quad \ell a r &= \operatorname{diag}(\alpha_-, \alpha_p, \alpha_+) = \begin{pmatrix} \alpha_- & 0 & 0 \\ 0 & \alpha_p & 0 \\ 0 & 0 & \alpha_+ \end{pmatrix}; \end{aligned}$$

with  $\alpha_p$  scalar,  $\alpha_-, \alpha_+$  symmetric square matrices (with dimensions  $p-1$  and  $n-p$  respectively),  $\alpha_- \leq -c < 0 < c \leq \alpha_+$ ; and

$$(37) \quad \operatorname{Re} \ell b r \geq -C\varepsilon.$$

*Proof.* Since  $a$  is symmetric, it is possible to find an orthonormal transformation  $\omega = \omega(W)$  such that  $\omega^t a \omega$  is (block-)diagonal with the decomposition given in the righthand side of (36). The spectral separation assumption guarantees the positivity/negativity of  $\alpha_+/\alpha_-$ . Moreover, the matrix  $\omega^t b \omega$  is positive semidefinite,  $\operatorname{Re} b \geq 0$ .

Let  $\omega_p$  denote the  $p$ th column of  $\omega$  and  $\gamma = \gamma(x)$  be the solution of the first order linear differential equation

$$(38) \quad \gamma' = -(\omega_p \cdot d\omega_p W') \gamma, \quad \gamma(0) = 1,$$

or, equivalently, set

$$(39) \quad \gamma(x) := \exp \left( \int_0^x \omega_p^t(W) d\omega_p(W) W' dy \right).$$

Define the matrix  $r$  and  $\ell$  as

$$(40) \quad r(x) := \omega(W) \operatorname{diag}(I_{p-1}, \gamma(x), I_{n-p}), \quad \ell(x) := r^{-1}(x).$$

Clearly estimates on  $|r'|$  and  $|\ell'|$  hold and

$$\begin{aligned} \ell a r &= \operatorname{diag}(I_{p-1}, \gamma^{-1}, I_{n-p}) \operatorname{diag}(\alpha_-, \alpha_p, \alpha_+) \operatorname{diag}(I_{p-1}, \gamma, I_{n-p}) \\ &= \operatorname{diag}(\alpha_-, \alpha_p, \alpha_+), \end{aligned}$$

hence  $\ell$  and  $r$  still block-diagonalize  $a$  in the manner claimed. Moreover

$$(\ell r')_{pp} = \gamma^{-1} \omega_p \cdot (\gamma \omega_p)' = \omega_p \cdot (d\omega_p W' - (\omega_p \cdot d\omega_p W') \omega_p) = 0,$$

since  $\omega_p$  has norm equal to 1.

By (39), it follows that  $\gamma = 1 + O(\varepsilon)$ ; hence bound (37) follows from assumptions on  $b$  and continuity.  $\square$

*Proof of Proposition 1.7.* To prove (15), it is sufficient to establish the corresponding result for  $\zeta := \ell z$  with  $\ell$  given in Lemma 3.2. Left multiplying (32) by  $\ell$ , we get the eigenvalue equation for  $\zeta$

$$(41) \quad \lambda \zeta + \alpha \ell r' \zeta + \alpha \zeta' - \beta \zeta'' = \Xi,$$

where

$$\alpha := \ell a r, \quad \beta := \ell b_0 r, \quad \Xi := \Xi_1 + \Xi_2'$$

and

$$\begin{cases} \Xi_1 := O(1)|W'|(\varepsilon \zeta + \zeta' + \hat{v}' + w'), \\ \Xi_2 := O(1)|W'|(\zeta + \hat{v}) + O(1)\varepsilon \zeta' + O(1)w'. \end{cases}$$

Following [4], set  $\rho_0(x) := 1$  for any  $x$ , and define the two weights  $\rho_{\pm}$  as the solutions to the Cauchy problem

$$(42) \quad \rho'_{\pm} = \mp M |W'| c^{-1} \rho_{\pm}, \quad \rho_{\pm}(0) := 1,$$

where  $c$  is given in Lemma 3.2 and  $M$  is a constant to be chosen later. Therefore, for  $\varepsilon$  so small that  $O(M\varepsilon) < 1$ ,

$$(43) \quad \rho_{\pm}(x) = \exp\left(\pm \int_0^x M |W'(\xi)| c^{-1} d\xi\right) = 1 + O\left(M \int_{\mathbb{R}} |W'(\xi)| d\xi\right) = O(1),$$

and

$$(44) \quad \rho'_j(x) = O(1) |W'(x)|, \quad j \in \{-, 0, +\}.$$

Let  $\rho = \rho(x)$  be the block diagonal matrix defined by

$$\rho(x) := \text{diag}(\rho_{-}(x) I_h, \rho_0(x), \rho_{+}(x) I_k)$$

where  $I_n$  denotes the identity  $n \times n$  matrix. Taking the real part of the  $L^2$ -scalar product of  $\rho \zeta$  against (41), we get

$$(45) \quad \text{Re} \lambda \langle \rho \zeta, \zeta \rangle + \text{Re} \langle \rho \zeta, \alpha \ell r' \zeta \rangle + \text{Re} \langle \rho \zeta, \alpha \zeta' \rangle - \text{Re} \langle \rho \zeta, \beta \zeta'' \rangle = \text{Re} \langle \rho \zeta, \Xi \rangle.$$

The weights  $\rho_0, \rho_{\pm}$  are positive and  $O(1)$ , hence  $\langle \rho \zeta, \zeta \rangle^{1/2}$  is equivalent to  $|\zeta|_{L^2}$ .

Both  $\rho$  and  $\rho \alpha$  are symmetric, hence

$$\text{Re} \langle \rho \zeta, \alpha \zeta' \rangle = \text{Re} \langle \zeta, \rho \alpha \zeta' \rangle = -\frac{1}{2} \text{Re} \langle \zeta, (\rho' \alpha + \rho d\alpha W') \zeta \rangle.$$

By (9) and (11), we have the key fact<sup>3</sup>

$$d\alpha_0 W' \leq -C |W'|$$

for some  $C > 0$ . By definition of  $\rho_{\pm}$ , we have also

$$\rho'_{\pm} \alpha_{\pm} + \rho_{\pm} d\alpha_{\pm} W' = \mp \rho_{\pm} (M |W'| c^{-1} \alpha_{\pm} - d\alpha_{\pm} W').$$

Thus, for  $M$  sufficiently large, there exists  $C > 0$ , independent on  $\varepsilon$ , such that

$$\begin{aligned} \rho' \alpha + \rho d\alpha W' &= \text{diag}(\rho'_- \alpha_- + \rho_- d\alpha_- W', d\alpha_0 W', \rho'_+ \alpha_+ + \rho_+ d\alpha_+ W') \\ &\leq -C |W'| \text{diag}(M, 1, M). \end{aligned}$$

---

<sup>3</sup>Indeed, this is what drives the Goodman estimate; see [3, 4].



Decomposing  $\zeta$  as  $(\zeta_-, \zeta_0, \zeta_+)$  and setting  $\hat{\zeta} := (\zeta_-, \zeta_+)$ , we get the “good” term

$$\operatorname{Re} \langle \rho \zeta, \alpha \zeta' \rangle \geq C \int_{\mathbb{R}} (M |\hat{\zeta}|^2 + |\zeta_0|^2) |W'| dx.$$

Next, let us deal with the “bad” term  $\langle \rho \zeta, \alpha \ell r' \zeta \rangle$ . Since  $(\ell r')_{pp} = 0$ , there holds

$$|\operatorname{Re} \langle \rho \zeta, \alpha \ell r' \zeta \rangle| \leq C \int_{\mathbb{R}} |\hat{\zeta}|^2 |W'| dx$$

Hence, by choosing  $M$  large enough, we get from (45)

$$\operatorname{Re} \lambda |\zeta|_{L^2}^2 + |\zeta|_{w'}^2 - \operatorname{Re} \langle \rho \zeta, \beta \zeta'' \rangle \leq C |\operatorname{Re} \langle \rho \zeta, \Xi \rangle|$$

Since

$$\operatorname{Re} \langle \rho \zeta, \beta \zeta'' \rangle = -\operatorname{Re} \langle \rho \zeta', \beta \zeta' \rangle - \operatorname{Re} \langle \rho \zeta, \beta' \zeta' \rangle - \operatorname{Re} \langle \rho' \zeta, \beta \zeta' \rangle$$

the term with  $\beta$  can be estimated by

$$\operatorname{Re} \langle \rho \zeta, \beta \zeta'' \rangle \leq C\varepsilon (|\zeta|_{w'}^2 + |\zeta'|_{L^2}^2)$$

having used (37). Hence, we obtain

$$(46) \quad \operatorname{Re} \lambda |\zeta|_{L^2}^2 + |\zeta|_{w'}^2 \leq C\varepsilon (|\zeta|_{w'}^2 + |\zeta'|_{L^2}^2) + C |\operatorname{Re} \langle \rho \zeta, \Xi \rangle|$$

Given  $\eta > 0$ , recalling (10), we deduce

$$|\operatorname{Re} \langle \rho \zeta, \Xi_1 \rangle| \leq C(\varepsilon + \eta) |\zeta|_{w'}^2 + C\eta^{-1}\varepsilon^2 (|\zeta'|_{L^2}^2 + |\hat{v}|_{L^2}^2 + |w'|_{L^2}^2)$$

with  $C$  independent on  $\eta$ . For what concerns the term with  $\Xi_2$ , integrating by parts and using (44), there holds

$$|\operatorname{Re} \langle \rho \zeta, \Xi_2' \rangle| = |\operatorname{Re} \langle \rho' \zeta, \Xi_2 \rangle| + |\operatorname{Re} \langle \rho \zeta', \Xi_2 \rangle| \leq |\operatorname{Re} \langle O(1) |W'| \zeta, \Xi_2 \rangle| + |\operatorname{Re} \langle \rho \zeta', \Xi_2 \rangle|$$

For any  $\eta > 0$ , estimating one by one the terms in  $\Xi_2$ , we obtain

$$|\operatorname{Re} \langle O(1) |W'| \zeta, \Xi_2 \rangle| \leq C(\varepsilon + \eta) |\zeta|_{w'}^2 + C\varepsilon^2 (|\zeta'|_{L^2}^2 + |\hat{v}|_{L^2}^2) + C\eta^{-1}\varepsilon^2 |w'|_{L^2}^2,$$

$$|\operatorname{Re} \langle \rho \zeta', \Xi_2 \rangle| \leq C\eta |\zeta|_{w'}^2 + C(\varepsilon + \eta^{-1}\varepsilon^2 + \eta) |\zeta'|_{L^2}^2 + C\eta^{-1}\varepsilon^2 |\hat{v}|_{L^2}^2 + C\eta^{-1} |w'|_{L^2}^2.$$

Choosing  $\eta = \varepsilon$  and summing up, we get

$$|\operatorname{Re} \langle \rho \zeta, \Xi_2' \rangle| \leq C\varepsilon (|\zeta|_{w'}^2 + |\zeta'|_{L^2}^2 + |\hat{v}|_{L^2}^2) + C\varepsilon^{-1} |w'|_{L^2}^2.$$

Inserting these estimates in (46), we get, for  $\varepsilon$  sufficiently small,

$$\operatorname{Re} \lambda |\zeta|_{L^2}^2 + |\zeta|_{w'}^2 \leq C (|\zeta'|_{L^2}^2 + \varepsilon |\hat{v}|_{L^2}^2 + \varepsilon^{-1} |w'|_{L^2}^2).$$

Since  $\zeta = \ell z$ , from the above estimate we deduce

$$\operatorname{Re} \lambda |z|_{L^2}^2 + |z|_{w'}^2 \leq C (\varepsilon |z'|_{L^2}^2 + \varepsilon |\hat{v}|_{L^2}^2 + \varepsilon^{-1} |w'|_{L^2}^2).$$

for  $\varepsilon$  sufficiently small. Recalling that  $z' = u$ , estimate (15) is proved.  $\square$

## APPENDIX A. STRUCTURAL HYPOTHESES

In this Appendix, we briefly discuss the structural hypotheses of the introduction, verifying the assertions of Remark 1.2 that A1–A2 and B1–B2 follow from conditions (i)–(ii) of the remark (i.e., partial simultaneous symmetrizability plus genuine coupling) together with the assumed structure  $Q = (0_n, q)$ .

**Lemma A.1.** *Let  $Q = (0_n, q)$ . Then, (i)–(ii) of Rmk.1.2 imply A1–A2 and B1–B2.*

*Proof.* These follow by more general results of Yong [10].<sup>4</sup> We give a proof for completeness. As all properties are coordinate-independent properties of the linearization about constant states, we may without loss of generality take  $A^0$  block-diagonal. For,  $TA^0$  is block-lower triangular for  $T$  block-upper triangular, whence  $TA^0T^*$  is symmetric block-diagonal, and a left symmetrizer for the system obtained by the change of coordinates  $w \rightarrow (T^*)^{-1}w$ ,  $A \rightarrow (T^*)^{-1}AT^*$ ,  $Q \rightarrow (T^*)^{-1}QT^*$ .

Observing that  $\tilde{A}^0 := (A^0)^{-1}$  is a right symmetrizer if  $A^0$  is a left symmetrizer, we obtain

$$\tilde{A}^0 w_t + \tilde{A} w_x = \tilde{Q} w,$$

where  $\tilde{A}^0$  is symmetric positive definite and block-diagonal,  $\tilde{A}$  is symmetric, and  $d\tilde{Q} = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{q} \end{pmatrix}$  symmetric with  $q < 0$ . (Note: the latter key fact follows by

$$d\tilde{Q} = (T^*)^{-1}dQT^*\tilde{A}^0,$$

the fact that  $T^*$  is block-lower triangular, and that the first block row of  $dQ$  by assumption vanishes.) Rewriting, we have

$$w_t + \bar{A} w_x = \bar{Q} w,$$

where

$$\bar{A} = \begin{pmatrix} (\tilde{A}_{11}^0)^{-1}\tilde{A}_{11} & (\tilde{A}_{11}^0)^{-1}\tilde{A}_{12} \\ (\tilde{A}_{22}^0)^{-1}\tilde{A}_{21} & (\tilde{A}_{22}^0)^{-1}\tilde{A}_{22} \end{pmatrix}, \quad d\bar{Q} = \begin{pmatrix} 0 & 0 \\ 0 & (\tilde{A}_{22}^0)^{-1}\tilde{q} \end{pmatrix}.$$

In these coordinates, one readily computes that

$$a_0 = (\tilde{A}_{11}^0)^{-1}\tilde{A}_{11}, \quad b_0 = -(\tilde{A}_{11}^0)^{-1}\tilde{A}_{12}^*\tilde{q}\tilde{A}_{12}^*,$$

hence  $\ker b_0 = \ker \tilde{A}_{12}$ , and genuine coupling for the reduced system is the condition that no eigenvector of  $a_0 = (\tilde{A}_{11}^0)^{-1}\tilde{A}_{11}$  lie in  $\ker \tilde{A}_{12}$ , the same condition as for

---

<sup>4</sup> Symmetrizability is not explicitly stated in [10], but is clear from the development.

genuine coupling of the full system, and  $\tilde{A}_{11}^0$  is a left symmetrizer for the reduced system with  $\tilde{A}_{11}^0 b_0 = -\tilde{A}_{12}^* \tilde{q} \tilde{A}_{12}^*$  symmetric positive semidefinite since  $\tilde{q}$  is symmetric negative definite.  $\square$

## REFERENCES

- [1] Broadwell J.E., *Shock structure in a simple discrete velocity gas*, Physics Fluids 7 (1964) no.8, 1243–1247.
- [2] Godillon P., Lorin E., *A Lax shock profile satisfying a sufficient condition of spectral instability*, J. Math. Anal. Appl. 283 (2003), 12–24.
- [3] Goodman, J., *Nonlinear asymptotic stability of viscous shock profiles for conservation laws*, Arch. Rational Mech. Anal. 95 (1986), no. 4, 325–344.
- [4] Goodman J., *Remarks on the stability of viscous shock waves*, in "Viscous profiles and numerical methods for shock waves" (Raleigh, NC, 1990), 66–72, SIAM, Philadelphia, PA, 1991.
- [5] Humpherys J., Zumbrun K., *Spectral stability of small-amplitude shock profiles for dissipative symmetric hyperbolic-parabolic systems*, Z. Angew. Math. Phys. 53 (2002), no. 1, 20–34.
- [6] Liu T.-P., Yu S.-H., *Boltzmann equation: micro-macro decompositions and positivity of shock profiles*, Comm. Math. Phys. 246 (2004), no. 1, 133–179.
- [7] Mascia C., Zumbrun K., *Pointwise Green's function bounds and stability of relaxation shocks*, Indiana Univ. Math. J. 51 (2002), no. 4, 773–904.
- [8] Mascia C., Zumbrun K., *Stability of large-amplitude shock profiles of general relaxation systems*, SIAM J. Math. Anal. 37 (2005), no. 3, 889–913.
- [9] Plaza R., Zumbrun K., *An Evans function approach to spectral stability of small-amplitude shock profiles*, Discrete Contin. Dyn. Syst. 10 (2004), no. 4, 885–924.
- [10] Yong W.-A., *Basic aspects of hyperbolic relaxation systems*, in "Advances in the theory of shock waves", 259–305, Progr. Nonlinear Differential Equations Appl., 47, Birkhäuser Boston, Boston, MA, 2001.
- [11] Yong W.-A., Zumbrun K., *Existence of relaxation shock profiles for hyperbolic conservation laws*, SIAM J. Appl. Math. 60 (2000) no.5, 1565–1575.